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Dedicated to the glowing memory of Sir James Lighthill

JAMES LIGHTHILL 1999 MEMORIAL PAPER

Scaling laws for turbulent wall-bounded shear flows at very large Reynolds numbers

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Abstract. Deep ideas of Sir James Lighthill concerning turbulent flows are discussed in the beginning. The scaling laws for large-Reynolds-number flows are presented in their historical development. The underlying hypotheses are discussed and compared with experiments. Special attention is given to non-universal Reynolds-number-dependent scaling laws that reveal the incomplete similarity of the flows. Recent results concerning scaling laws for boundary layers are presented in more detail and discussed.

Key words: turbulence, scaling laws, wall-bounded shear flows, asymptotics, Reynolds number

1. Introduction

Sir James Lighthill's life in science created an unsurpassed standard for applied mathematicians. He was a legend in his lifetime since his early twenties. His works and his legendary image will continue to influence applied mathematics in the future and this influence will grow with time. As Milton Van Dyke, an outstanding applied mathematician and fluid dynamicist by his own, said: The generation of applied mathematicians who were lucky to live in Lighthill's time and to be directly influenced by Sir James will be known as the Lighthill generation.

There are now wide and sometimes fervant discussions concerning the subject of applied mathematics, and the role and responsibilities of applied mathematicians. It seems to the present author that the key to the correct answer to these questions lies in the famous saying of J. W. Gibbs: 'Mathematics is also a language'.

Indeed, all people use language. However, among the users of language, a particularly important group can be distinguished: authors – poets, novelists, etc. They create fictitious images, paradigms – idealized models of people and social phenomena. The greatest of these paradigms, like Francesca da Rimini, Romeo and Juliet, Dr. Faust, Anna Karenina and the circumstances that surround them, continue to live for centuries. They transform culture, and in particular language.

A similar role is played by applied mathematicians. Using the language of mathematics and transforming it when necessary, applied mathematicians create models of phenomena, both in nature and engineering. These models give idealized but sufficiently complete images of phenomena as a whole, which allow their mathematical analysis. The purpose of these models is to predict the behavior of systems in unexplored ranges. When this goal is achieved, it leads to practical applications. To be a dedicated applied mathematician is a great achievement, honour and privilege, and Sir James proved it by his life in science.

Sir James Lighthill worked in applied mathematics for more than 50 years. He created new branches of fluid mechanics which will be leading in the coming XXI century. He elucidated seemingly frozen classical fields in fluid mechanics from unexpected new viewpoints and returned them to active life. He was equally strong in creating new models and in inventing mathematical tools for exploring models of new challenging phenomena.

The present paper is concerned with hydrodynamic turbulence. The study of turbulence is a singular branch of fluid mechanics, and the contribution of Sir James to this field is very significant, however, rather astonishingly, not fully appreciated.

His widely known papers [1, 2] concerning aerodynamically generated sound and the role of turbulence in sound generation are repeatedly reviewed, also by Sir James himself. However, his survey 'Turbulence' [3] (published by Manchester University Press in the volume dedicated to Osborne Reynolds) remains not as well known as it should be, while it is truly remarkable. It is now available in a four-volume edition of Sir James Lighthill's Collected Papers, published by Oxford University Press and edited by Professor M. Y. Hussaini [4], and all devoted fluid dynamicists are able now to enjoy reading it. (Thanks to Oxford University Press and Professor Hussaini!) Sir James was able to present in this paper practically all essential ideas of this subject and critically evaluate them. And it was done in only 64 pages – that is the standard of how turbulence should be presented! As the great Russian poet, Alexandre Block, said, 'Rub out the accidental features, and you will see – the world is marvellous' (English translation by Sir James Lighthill). In writing this paper the present author was truly inspired by this remarkable survey.

The paper [5], 'Effect of compressibility on turbulence' also remains practically unknown to fluid dynamicists working in turbulence, and it is now perfectly clear why. It was published as one of the chapters in the volume *Gas Dynamics of Cosmic Clouds.* But it did not contain any applications to astrophysical problems, and therefore did not attract due attention of astrophysicists. Also, fluid dynamicists not related to astrophysics did not have the idea of looking in this volume, even though this paper contains a constructive model of turbulence in compressible fluid at sufficiently large Mach numbers. It is demonstrated there that under such conditions turbulence is created not only by vortices, but also by peculiar generalized N-shaped shock waves. Everything is ready for an application of the invariance relations! Sir James emphasized at the end of this short paper:

> The author feels rather that the system (*at Mach numbers comparable with one or greater and in the three-dimensional case – GB*) has become one in which the division of the motion into 'turbulence' on the one hand and 'sound' (or shock waves) on the other is almost without significance.

Great words of general significance! Exactly the same situation happens with turbulence in stratified fluids; the role of Mach number is played here by the Richardson number. Both phenomena should be considered as a unified new phenomenon rather than as a modification of turbulence in an incompressible homogeneous fluid.

The last paper of Sir James closely related to turbulence was recently published in this Journal [6]. It contains a crucial new 'sandwich' model of a hurricane. Sir James writes:

> Between ocean and atmosphere there exists at high wind speeds a thick layer of 'a third fluid': ocean spray consisting of a relatively tall cloud of droplets. Many of the smaller ones… appear where air bubbles burst at the sea surface. A greater mass of droplets, on the other hand, is formed… as 'splash' torn from, or as 'spume' ejected from whitecaps (in the form of droplets with radii ranging from about 20 μm to much larger values).

That is to say, in this intermediate layer appears the basic reason for the phenomenon – the damping of turbulence. Because the work spent by turbulence to keep the droplets suspended is taken from the kinetic energy of the turbulence, the turbulent energy is reduced. The role of large droplets, whose substantial number in the third fluid layer Sir James particularly emphasized, is of great importance, because the large droplets prevent suspension of all the droplets by wind. The large droplets leave the 'third fluid' in a rather thin layer. The turbulent drag and heat exchange in the 'third fluid layer' are substantially reduced, and this layer works as a lubricator and thermal insulator for the wind, increasing its velocity. The qualitative theory suggested by Lighthill's 'sandwich' model is now under development. When Lighthill's model is confirmed and brought to a more quantitative shape *it will open practical possibilities of preventing or at least suppressing hurricanes.*

It is appropriate here to give a definition of hydrodynamic turbulence. *Turbulence is the state of vortex fluid motion where the velocity, pressure, and other properties of the flow field vary in time and space sharply and irregularly and, it can be assumed, randomly.* Turbulent fluid flows surround us, in the atmosphere, the oceans, in engineering and biological systems. First recognized and examined by Leonardo, for the past century turbulence has been studied by engineers, mathematicians and physicists, including such giants as Kolmogorov, Heisenberg, Taylor, Prandtl, and von Kármán. Every advance in a wide collection of subjects, from chaos and fractals to field theory, and every increase in the speed and parallelization of computers is heralded as ushering in the solution of the 'turbulence problem', yet turbulence remains the greatest challenge of applied mathematics as well as of classical physics.

It is very discouraging that, in spite of hard work by an army of scientists and research engineers during more than a century, almost nothing has become known about turbulence from first principles: the continuity equation and the Navier-Stokes equations:

$$\partial_\alpha u_\alpha = 0, \tag{1}$$

$$\partial_t u_i + \partial_\alpha u_i u_\alpha = -\frac{1}{\rho}\partial_i p + \nu \Delta u_i. \tag{2}$$

(Here standard notations are used: the u_i ($i = 1, 2, 3$) are the velocity components in a rectilinear orthonormal Cartesian coordinate system x_1, x_2, x_3, p is the pressure, t is the time, $\partial_i = \partial/\partial x_i$, Δ is the Laplacian, ν is the kinematic viscosity, and ρ is the density; repeated Greek indices imply summation.)

Turbulence at very large Reynolds numbers (often called developed turbulence) is widely considered to be one of the happier provinces of the turbulence realm, as it is widely thought that two of its basic results are well-established, and have a chance to enter, basically untouched, into a future complete theory of turbulence. These results are the von Kármán–Prandtl universal logarithmic law in the wall-region of wall-bounded turbulent shear flow, and the Kolmogorov–Obukhov scaling laws for the local structure of developed turbulent flow. The beginning of fundamental research of turbulent flows at very large Reynolds numbers can be dated sharply: it was the lecture of Th. von Kármán at the Third International Congress for Applied Mechanics at Stockholm, 25 August 1930 [7]. Von Kármán, one of the greatest applied mechanicians of the XX century was the principal founder of the International Congresses for Applied Mechanics.[1] Unquestionably his lecture 'Mechanische Ähnlichkeit und Turbulenz'[2] was the central event of the Congress.

[1] These Congresses continue regularly, once every four years, up to the present time under the name International Congresses for Theoretical and Applied Mechanics.

[2] Mechanical Similitude and Turbulence.

Von Kármán began his lecture with the following statement:

> Unsere experimentellen Kenntnisse über die innere Struktur der turbulenten Strömung genügen noch nicht, um sichere Grundlagen für eine rationelle theoretische Berechnung der Geschwindigkeitsverteilung und der Reibung im sogenannten hydraulischen Strömungszustande zu liefern. Die zahlreichen halbempirischen Ansätze, z. B. der Versuch einen turbulenten Reibungskoeffizienten einzuführen, können weder den Theoretiker noch den Praktiker befriedigen. Auch die nachfolgenden Untersuchungen erheben nicht den Anspruch, eine wirkliche endgültige Theorie der Turbulenz zu liefern. Ich will mich vielmehr darauf beschränken, klarzustellen, was man auf Grund der reinen Hydrodynamik auszusagen vermag, wenn man über bestimmte grundlegende Fragen bestimmte Hypothesen einführt. [3]

The hypothesis proposed by von Kármán for answering the fundamental questions concerning the velocity distributions and drag coefficient in turbulent hydraulic, or shear, flows as they are called now, first of all flows in pipes and channels, was presented by him in the following straightforward form:

> Wir gründen auf diese experimentell festgestellten Tatsachen die Annahme, dass, abgesehen von der Wandnähe, die Geschwindigkeitsverteilung der mittleren Strömung von der Zähigkeit unabhängig ist.[4]

As a result of subsequent arguments proposed by von Kármán there appeared what is called now the universal (Reynolds-number-independent) logarithmic law and corresponding drag law for the turbulent flow in a cylindrical pipe.

The leaders of applied mechanics of that time were present at von Kármán's lecture and took part in the subsequent discussion. The first speaker was L. Prandtl. He said:

> **Herr Prandtl:** Die neuen Kármán'schen Rechnungen bedeuten einen höchst erfreulichen Fortschritt in der Frage der Flüssigkeitsreibung. Bisher war es immer so, dass beim Fortschreiten in den höheren Reynolds'schen Zahlen die frühere Interpolationsformel sich bei der Extrapolation auf das neu erforschte Gebiet als unrichtig erwies und durch eine neue ersetzt werden musste. Die Forschungslaboratorien machten grosse Anstrengungen, immer höhere Reynolds'schen Zahlen zu erreichen, doch setzten die Kosten der grossen Versuchseinrichtungen eine Grenze, die kaum mehr überschritten werden konnte.
>
> Durch die Kármán'schen Formeln sind nun weitere Anstrengungen in dieser Richtung unnötig geworden. Die Formeln sind sowohl mit den Rohrversuchen von Nikuradse und von Schiller und Hermann und mit den Plattenreibungsversuchen von Kempf in so gutem Einklang, dass man ihnen alles Vertrauen für Anwendung auf beliebig hohe Reynolds'schen Zahlen schenken darf. Nach kleineren Reynolds'schen Zahlen hin ist die Übereinstimmung schlechter, was auf eine Zähigkeitswirkung auch im Innern der Flüssigkeit zurück-

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[3] Our experimental knowledge of the internal structure of turbulent flows is insufficient for delivering a reliable foundation for a rational theoretical calculation of the velocity distribution and drag in the so-called hydraulic flow state. Numerous semi-empirical formulae, for instance, the attempt to introduce turbulent drag coefficients, are unable to satisfy neither the theoretician nor the practitioner. The investigations which will be presented below also do not claim to achieve a genuine ultimate theory of turbulence. I will restrict myself rather to clarifying what can be achieved on the basis of pure fluid dynamics if definite hypotheses are introduced concerning definite basic questions.

[4] On the basis of these experimentally well established facts we make an assumption that outside a close vicinity of the wall the velocity distribution of the mean flow is viscosity-independent.

zuführen ist, d. h. der von der Zähigkeit beeinflusste Streifen, der keineswegs bloss aus der Laminarschicht an der Wand besteht, reicht hier weit ins Innere herein.

Bezüglich der beiden Darstellungen der Geschwindigkeitsverteilung bei Nikuradse im Anschluss an die neuen Kármán'schen Formeln und an meine frühere Formulierung mit dem dimensionslos gemachten Wandabstand möchte ich noch auf einen scheinbaren Widerspruch hinweisen. Die Kármánschen Formeln berücksichtigen die Zähigkeit nur als eine Randbedingung. Die Geschwindigkeitsverteilung wird dort ohne Zähigkeit berechnet. Der dimensionslose Wandabstand $y^* = (y/\nu)\sqrt{\tau_0/\rho}$ enthält aber die Zähigkeit. Meiner Ansicht nach ist die Aufklärung die, dass für die ganz grossen Reynoldsschen Zahlen die Kármánschen Darstellung als das Exakte aufzufassen ist, während die Darstellung mit dem dimensionslos gemachten Wandabstand wesentlich den Wandstreifen genau wiedergibt, in dem die Zähigkeit mit der Turbulenz zusammen wirkt.[5]

It should be understood that at that time L. Prandtl was generally considered as 'the chief of applied mechanicians' (*cf*. Batchelor [8], p. 185). The opinion which we just reproduced explains at least partially why during nearly seventy years the Nikuradze [9] experiments[6] were never extended to larger Reynolds numbers. And, moreover, the culture of such experiments, in fact very subtle, decayed and to a certain extent was lost.

It is also true that the last part of L. Prandtl's comment is very deep and instructive. But it remained dormant and was not cast into a proper mathematical theory for the following technical reason. In the early thirties, and even long before, the mathematical techniques which were needed here were in sufficiently good shape. However, they were considered as something like a mathematical monstrosity with no practical applications. Only several decades later it was recognized (see [10–13]) that many physical phenomena needed these techniques for modeling, and they entered the practice of applied mathematics and theoretical physics as incomplete similarity, fractals, renormalization groups. These concepts will be used in the present paper to explain the situation with the scaling laws for turbulent shear flows at very large Reynolds number. In particular, incomplete similarity will allow the resolution of the contradiction mentioned in the last part of Prandtl's comment.

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5 The new Kármán calculations signify a very enjoyable progress in the problem of fluid friction. It was always the case, that by advancing to higher Reynolds numbers the previous interpolation formulas were revealed to be incorrect by extrapolation to a newly investigated range, and had to be replaced by new ones. Research laboratories made big efforts to achieve higher Reynolds numbers, but the cost of big experimental set-ups has some bound which cannot be substantially exceeded. *Due to Kármán's formulas further efforts in this direction became unnecessary.* [italics mine – GB] The formulas are in such good agreement with the experiments in pipe flows by Nikuradze, and of Schiller and Hermann, and with the experiments concerning the drag of plates performed by Kempf, that complete confidence can be placed in them for their application at arbitrarily large Reynolds numbers. For lower Reynolds numbers the agreement is worse, and this can be attributed to the action of the viscosity also in the inner part of the flow, *i.e.* to the viscosity-influenced streaks of which the laminar layer at the wall consists, and which in this case enter far into the internal part of the flow.

I want to point out a seeming contradiction concerning both representations of the velocity distribution by Nikuradze in connection with Kármán's new formulas and my earlier formulation using the dimensionless distance from the wall. Kármán's formulas use viscosity in the boundary condition only. The velocity distribution should be calculated without viscosity. However, the dimensionless distance from the wall, $y^* = (y/\nu)\sqrt{\tau_0/\rho}$, does contain the viscosity. According to my opinion, the explanation is that the Kármán representation should be considered as exact for very large Reynolds numbers, while the representation via the dimensionless distance from the wall applies essentially to the wall layer and streaks where the viscosity and turbulence are acting together.

6 According to the author's opinion, Nikuradze's experiments, as far as the flows in smooth pipes are concerned, are adequate in their range of Reynolds number except at the lowest ones.

After von Kármán's [7] work, L. Prandtl [14] also came to the universal logarithmic law using a different approach, and the name 'von Kármán–Prandtl universal logarithmic law' became established. Many different derivations of the universal logarithmic law were proposed later (see Sir James Lighthill's survey [3], pp. 116–117; Landau and Lifshitz [15], pp. 172–175; Schlichting [16] pp. 489–490; Monin and Yaglom [17], pp. 273–274, and quite recently, Spurk [18], p. 231). We emphasize, however, that the basis of all these derivations remained the hypothesis explicitly formulated by von Kármán [7] which was cited earlier.

The second major breakthrough in the theory of turbulence at very large Reynolds numbers happened in 1941 in the fundamental works of A. N. Kolmogorov and A. M. Obukhov, at that time Kolmogorov's student [19–22], where the laws of the local structure of such flows were obtained. We emphasize particularly the role of the elucidating paper by G. K. Batchelor [23] in which the Kolmogorov–Obukhov theory, presented originally in the form of short notes, was explained in detail and fundamentally clarified. The problems of the local structure of developed turbulence are, however, outside the scope of the present paper.

One important note in conclusion. Early in 1996 the present author came to Berkeley for a short visit and had the privilege of meeting Professor A. J. Chorin. During our first conversation we discovered that we had been working on similar problems with different but complementary tools, which, when wielded in unison, led to unexpected results. We have been working together ever since, and it is my great pleasure to present in this paper dedicated to Sir James Lighthill some of the results of our joint work. The contributions of Dr. V. M. Prostokishin and Professors N. D. Goldenfeld and O. Hald are appreciated by both of us.

2. Scaling and incomplete similarity

The short explanation of these concepts will be presented here for the reader's convenience and exactly in the form in which it will be used in the present paper. More detailed and general explanations can be found in the books [11–13].

Consider a physically meaningful relation

$$y = f(x_1, x_2, x_3, c), \tag{3}$$

where the arguments x_1, x_2, x_3 have independent dimensions, like, for example, density, energy and time, and the dimensions of y and c are dependent ones, *i.e.*, they can be expressed as

$$\begin{aligned} [y] &= [x_1]^p [x_2]^q [x_3]^r \\ [c] &= [x_1]^k [x_2]^\ell [x_3]^m . \end{aligned} \tag{4}$$

We consider here mechanical phenomena only, so the number of arguments having independent dimensions is no more than three. The symbol $[x]$ denotes the dimension of the quantity x, and we restrict ourselves here to the case of a single argument c having dependent dimensions, needed in the present paper.

The relation (3) is physically meaningful; therefore dimensional analysis allows one to represent the relation (3) in the form

$$\Pi = \Phi(\Pi_1), \tag{5}$$

where

$$\Pi = \frac{y}{x_1^p x_2^q x_3^r} \qquad \Pi_1 = \frac{c}{x_1^k x_2^\ell x_3^m}, \tag{6}$$

are dimensionless quantities, and Φ is an unspecified function. This means that the function f has the important property of generalized homogeneity

$$f(x_1, x_2, x_3, c) = x_1^p x_2^q x_3^r \Phi\left(\frac{c}{x_1^k x_2^\ell x_3^m}\right).$$

Consider now the case when the quantity Π_1 is very small, $\Pi_1 \ll 1$, or very large $\Pi_1 \gg 1$. In such cases, in the practice of mathematical modeling, it is customary to assume that the function Φ (Π_1) can be replaced by a constant, *i.e.*, by its limit C at $\Pi_1 \to 0$ or $\Pi_1 \to \infty$. Indeed, if the function Φ has a finite nonzero limit C and Π_1 is small (or large) enough, this is true, and it is possible to replace Equation (3) by a much simpler relation

$$y = Cx_1^p x_2^q x_3^r\,. \tag{7}$$

Thus, in such cases, (i) the parameter c disappears completely, and (ii) the powers p, q and r can be found by a simple algebraic procedure. A classical example of such a situation is the very intense explosion, investigated by G. I. Taylor and J. von Neumann (see, *e.g.* [13], pp. 47–49). When such a situation holds one says that there is a *complete similarity* in the parameter Π_1.

It is obvious that, in general, complete similarity does not hold: *there is, in general, no reason to believe that for all physical phenomena the function* Φ *has a finite nonzero limit when* Π_1 *goes to zero.*[7] Therefore, the parameter Π_1 generally speaking remains essential, even when it is small, and thus the argument c in the relation (3) remains essential. This general statement is rather trivial, and nothing particularly significant can follow from it.

There exists, however, an important intermediate special case which reveals a wide class of scaling relations in many important phenomena. This is the case when the function Φ has no finite nonzero limit when Π_1 goes to zero or infinity, but at small (large) Π_1 can be represented as

$$\Phi(\Pi_1) = C\Pi_1^\alpha + \cdots, \tag{8}$$

where C and α are constants, and the dots represent terms smaller than the first one. In addition, we have to remember that we are interested not in the limit, but in the asymptotics, more precisely, intermediate asymptotics for Π_1 small or large. Neglecting smaller terms, as is possible for sufficiently small (large) Π_1 and substituting (8) in (5), we obtain for small (large) Π_1:

$$\Pi = C\Pi_1^\alpha \tag{9}$$

or, returning to dimensional variables,

$$y = Cx_1^{p-\alpha k} x_2^{q-\alpha\ell} x_3^{r-\alpha m} c^\alpha\,. \tag{10}$$

[7] Another example of such widely believed legends is the naive expectation that the constant C should be of the order of one. A brilliant discussion of a case where this is not so and why can be found in the paper by Sir James Lighthill [5], where the effects of compressibility on turbulence are considered.

This power ('scaling') relation is of the same form as (7), however, with two essential distinctions. First, while the powers of the variables x_1, x_2, x_3 were easily obtained in the case of complete similarity (7) by dimensional analysis, in the relation (10) they cannot be so obtained because the parameter α which enters the relation (10) is a particular property of the problem under consideration and its value cannot be obtained from the general covariance principle which is the basis of dimensional analysis. Therefore its determination requires an effort beyond dimensional analysis. Furthermore, contrary to (7), the argument c does not disappear from the resulting scaling relation, but enters this relation in a power combination with other governing parameters. We refer to such cases as cases of *incomplete similarity in the parameter* Π_1.

Scaling relations with powers that cannot be obtained from dimensional considerations have a long history in engineering. A widely shared opinion held, and very often holding now, is that these relations are nothing more than empirical correlations. In fact, these relations are especially important because they also reveal the self-similarity of the phenomena, but a more complicated case of it. We shall see below that this is exactly what happens in turbulence at very high Reynolds number. Here viscosity has a persistent effect, despite very large values of the corresponding dimensionless parameter, the Reynolds number. It does not disappear, but it enters the resulting relations only in combination with other parameters of the turbulent flow.

3. Mathematical example

Chorin proposed a remarkable mathematical example which elucidates the nontrivial mathematical situation in the problem of turbulent shear flows at large Reynolds numbers. Chorin's example can be compared by its value with the example of Friedrichs which elucidated boundary layer theory ([24], pp. 155–156, [25], pp. 69–70).

Consider a family of curves

$$\phi = \left(\log \frac{d}{\delta}\right)\left(\frac{y}{\delta}\right)^{1/\log(d/\delta)} - 2\log\frac{d}{\delta}, \tag{11}$$

where ϕ is a dimensionless function, d and δ are parameters with the dimension of length, y is the independent variable, also having the dimension of length, $y > \delta$. We assume d is fixed and δ is the parameter of the family.

It is easy to show that the function ϕ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = \left(\frac{1}{\log\left(\frac{d}{\delta}\right)} - 1\right)\frac{1}{y}\,\frac{\mathrm{d}\phi}{\mathrm{d}y} \tag{12}$$

and the boundary conditions

$$\phi(\delta) = -\log\frac{d}{\delta}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}y}\Big|_{y=\delta} = \frac{1}{\delta}. \tag{13}$$

Assume now that d is much larger than δ, $d \gg \delta$, so that $1/\log(d/\delta)$ is a small parameter. For the curves of the family (11) a simple relation is easily obtained

$$y\partial_y\phi = \left(\frac{y}{\delta}\right)^{1/\log(d/\delta)} = \exp\left[\frac{\log\frac{y}{d} + \log\left(\frac{d}{\delta}\right)}{\log\frac{d}{\delta}}\right]. \tag{14}$$

This relation shows that at $d/\delta \to \infty$ and any fixed y/d the quantity $y\partial_y\phi = \partial_{\log y}\phi$ tends to e.

As a function of δ the family (11) has an envelope

$$\phi = \log \frac{y}{d}. \tag{15}$$

The quantity $\partial_{\log y}\phi$ for the envelope is also a constant, but a different one, equal to 1. (We emphasize that here we consider only the branch of the family (11) having $d > \delta$. There is another branch with $d < \delta$ which also has an envelope $\phi = 2\log(y/d)(2-z)^{-1}$, where $z = 15936\ldots$ is the second, nonzero root of the equation $(2-z)\exp z = 2$.)

Assume now, that in Equation (12), *i.e.*, for $y > \delta$ but not in the boundary condition at $y = \delta$ we neglect (remember von Kármán's basic hypothesis!) the small parameter $1/\log(d/\delta)$ in comparison with 1, so that Equation (12) is reduced to the form

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = -\frac{1}{y}\frac{\mathrm{d}\phi}{\mathrm{d}y}. \tag{16}$$

Satisfying the (δ-dependent!) boundary conditions (13), we obtain not the family (11), but only a single curve, the envelope (15) which is δ-independent ('universal'!). In fact, by neglecting the small parameter $1/\log(d/\delta)$ in Equation (12) we prevented (compare with Prandtl's comment in the Introduction) the penetration of the influence of the parameter δ into the basic region.

Let us look at this matter from a different viewpoint. The derivative $\mathrm{d}\phi/\mathrm{d}y$ can be represented without solving Equation (12) and by dimensional analysis only in the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \frac{1}{y}\Phi\left(\frac{y}{\delta}, \frac{d}{\delta}\right), \tag{17}$$

where Φ is a function of its dimensionless arguments. In the case under consideration, we have

$$\Phi = \left(\frac{y}{\delta}\right)^{1/\log(d/\delta)}. \tag{18}$$

We see that, at arbitrarily large y/δ, the function Φ cannot be replaced by a constant, so that the influence of δ is preserved and cannot be neglected. However, δ enters the resulting equations in a specific, power-type form, due to specific type of invariance of the problem on the whole. In fact, we met in this example the incomplete similarity in the parameter (y/δ), explained in the previous section. We will see that the same situation happens in wall-bounded turbulent shear flows. However, if we do make the assumption of complete similarity, $\Phi =$ constant as above, then we recover the envelope of the family of solutions rather than the solutions themselves.

4. Steady shear flows at very large Reynolds numbers. The intermediate region in a pipe flow

We return to the problem of statistically steady turbulent shear flows[8] ('hydraulic flows' in von Kármán terminology). Among such flows are many flows of practical importance, such

[8] Shear flows are flows with parallel mean velocities varying only in the lateral direction.

as flows in pipes, channels and boundary layers. Their educational value is in fact related to their locality. In general, turbulent flows are nonlocal both in time and space, so that their mean properties are determined not only by the flow state at a given point, but also by the flow history and the flow properties at neighboring points. This is not so for steady turbulent shear flows, and the locality simplifies their study essentially. Flows in cylindrical pipes constitute an instructive example of wall-bounded turbulent shear flows.

We have the same clear goal and well-determined problems as once formulated by von Kármán: To obtain the mathematical expressions for the drag coefficient and the velocity distribution in the intermediate region of the flow. 'Intermediate' means outside the viscous sublayer adjacent to the wall where the velocity gradients are so high that the viscous stress is comparable with the stress created by turbulent vortices, and not too close to the pipe axis. Von Kármán also considered the same intermediate region of flow.

However, our basic hypothesis will be essentially different from von Kármán's hypothesis, presented in the Introduction, and this difference will lead to substantially different results. In fact, we replace von Kármán's hypothesis of complete similarity by that of incomplete similarity.

We turn to the derivation of the velocity distribution in the intermediate region. Assume that the mean velocity gradient $\partial_y u$ depends on the following arguments: the transverse coordinate y (the distance from the wall), the shear stress at the wall τ, the pipe diameter d, and the fluid properties: its kinematic viscosity ν and density ρ. The velocity gradient $\partial_y u$ is considered rather than the velocity u itself, because the values of u at an arbitrary distance from the wall depend on the flow in the vicinity of the wall where the asymptotic assumptions which we shall use are clearly invalid. Thus,

$$\partial_y u = f(y, \tau, d, \nu, \rho). \tag{19}$$

Following von Kármán and Prandtl we introduce the viscous length scale

$$\delta = \frac{\nu}{u_*}, \qquad \text{where} \quad u_* = \sqrt{\frac{\tau}{\rho}}, \tag{20}$$

and a rigorous application of dimensional analysis gives

$$\partial_y u = \frac{u_*}{y}\Phi\left(\frac{y}{\delta}\,,\ \frac{d}{\delta}\right). \tag{21}$$

Also, dimensional analysis shows that $d/\delta = u_* d/\nu$ is a function of the traditional Reynolds number

$$\mathrm{Re} = \frac{\bar{u}d}{\nu}, \tag{22}$$

where $\bar{u}$ is the average velocity which is the flux divided by the cross-section area. Von Kármán used the Reynolds number based on the maximum velocity. In principle, it makes no difference. The relation (21) can be rewritten therefore as

$$\partial_y u = \frac{u_*}{y}\Phi\left(\frac{y}{\delta}\,, \mathrm{Re}\right). \tag{23}$$

For very large Reynolds numbers, in the intermediate region under consideration, the ratio of the distance from the wall to the viscous length scale y/δ is large. The basic von Kármán hypothesis (see the Introduction) is that the viscosity does not affect the velocity distribution. However, viscosity enters both arguments y/δ and Re. Therefore this hypothesis means, in the terms of Section 2, complete similarity in the parameters y/δ and Re. According to von Kármán's hypothesis the viscous length scale δ should disappear from the resulting relations, and the function Φ can be replaced by a constant: $\Phi = 1/\kappa$. The constant κ was later named 'Kármán's constant'. Substitution of $\Phi = 1/\kappa$ in (23) gives

$$\partial_y u = \frac{u_*}{\kappa y}. \tag{24}$$

Integration gives the von Kármán–Prandtl universal (Reynolds-number independent) logarithmic law for the velocity distribution

$$u = u_* = \left[\frac{1}{\kappa}\log\frac{u_* y}{\nu} + C\right], \tag{25}$$

where the constant C (and this is also a seemingly logically consistent, but nevertheless additional substantial assumption) is finite and Re-independent. L. Prandtl (see his comment in the Introduction) emphasized a 'seeming contradiction' related to the appearance of the viscosity in the resulting formula, which will be explained later.

For more than six decades the experimental information accumulated, suggesting some doubts in the universal logarithmic law, *i.e.*, in the von Kármán basic hypothesis which we now call the hypothesis of complete similarity. The experimental data demonstrated a systematic deviation (not a scatter!) from the predictions of the universal logarithmic law even if a very liberal approach to the constants κ and C is allowed (κ from 0·38 to 0·44; C from 4·1 to 6·3!), although, by the very logic of the derivation, these constants should be identical for all high-quality experiments in smooth pipes. Therefore, it was a natural step for us to assume that there is no complete similarity, and to propose, instead of the von Kármán hypothesis, a different hypothesis, suggesting the next by complexity step:

First Hypothesis: *There is an incomplete similarity of the average velocity gradient in the parameter* y/δ, *and no kind of similarity in* Re.

According to this hypothesis, the influence of the viscosity remains at arbitrary large Reynolds numbers in the whole body of the flow, but the viscosity enters only in combination with other parameters controlling the turbulence. Practically, this means that for very large Re the function Φ in (23) at large y/δ should be assumed to be a power function of its argument y/δ, while no special suggestion of any kind of similarity in Re is assumed, so that

$$\Phi\left(\frac{y}{\delta}, \mathrm{Re}\right) = A(\mathrm{Re})\left(\frac{y}{\delta}\right)^{\alpha(\mathrm{Re})}, \tag{26}$$

where $A(\mathrm{Re})$ and $\alpha(\mathrm{Re})$ are certain, yet undetermined functions of the Reynolds number. It is instructive at this point to remember Chorin's example presented in Section 3.

Substituting (26) in (23) we obtain

$$\partial_y u = \frac{u_*}{y} A(\mathrm{Re})\left(\frac{y}{\delta}\right)^{\alpha(\mathrm{Re})}, \qquad \delta = \frac{\nu}{u_*}. \tag{27}$$

Note that the relation (24) is a special case of (27). Therefore, if the experiments and/or numerical computations (for the relevant range of high Reynolds numbers the numerical computations are nowadays impossible, so here we speak about the rather far future) would

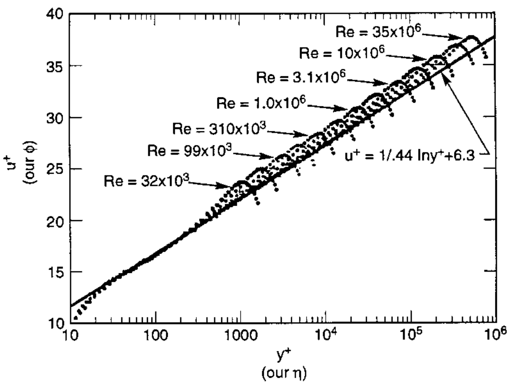

Figure 1. The Princeton data [27] obtained in a high-pressure pipe confirm the splitting of the experimental data according to their Reynolds numbers and the rise of the curves above their envelope in the $(\log\eta, \phi)$-plane. The solid line is the envelope; the curves turn at the center of the pipe. The splitting and form of the curves agree with the scaling law, and are incompatible with the von Kármán–Prandtl universal logarithmic law. Our interpretation of these experimental results is substantially different from that of the authors of [26]. Moreover, as we showed in [31], the results of [26, 27] at $\mathrm{Re} > 10^6$ are influenced by the roughness of the pipe wall. However, the splitting of experimental data is a very robust phenomenon clearly revealed in spite of experimental error. (Reproduced with permission from [26]).

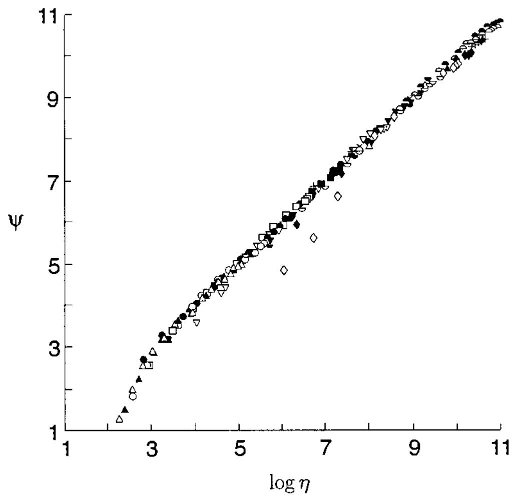

Figure 2. The experimental data of Nikuradze [9] in the coordinates $(\log\eta, \psi)$ at $\eta > 30$ lie close to the bisectrix of the first quadrant, confirming the scaling law. (1) △, $\mathrm{Re} = 4 \times 10^3$; (2) ▲, $\mathrm{Re} = 6{\cdot}1 \times 10^3$; (3) ○, $\mathrm{Re} = 9{\cdot}1 \times 10^3$; (4) ●, $\mathrm{Re} = 1{\cdot}67 \times 10^4$; (5) □, $\mathrm{Re} = 2{\cdot}33 \times 10^4$; (6) ■, $\mathrm{Re} = 4{\cdot}34 \times 10^4$; (7) ▽, $\mathrm{Re} = 1{\cdot}05 \times 10^5$; (8) ▼, $\mathrm{Re} = 2{\cdot}05 \times 10^5$; (9) ∪, $\mathrm{Re} = 3{\cdot}96 \times 10^5$; (10) ⬬, $\mathrm{Re} = 7{\cdot}25 \times 10^5$; (11) ◇, $\mathrm{Re} = 1{\cdot}11 \times 10^6$; (12) ◆, $\mathrm{Re} = 1{\cdot}536 \times 10^6$; (13) +, $\mathrm{Re} = 1{\cdot}959 \times 10^6$; (14) ×, $\mathrm{Re} = 2{\cdot}35 \times 10^6$; (15) ∩, $\mathrm{Re} = 2{\cdot}79 \times 10^6$; (16) ☗, $\mathrm{Re} = 3{\cdot}24 \times 10^6$.

The application of the same processing to the Princeton data [26, 27] as well as the analysis of the drag curves revealed [31] the influence of wall roughness at $\mathrm{Re} > 10^6$.

show that A is a universal constant while $\alpha = 0$, we could return to (24). *Now we can claim definitely that this is not the case!*

Note immediately a clear-cut qualitative difference between the cases of complete and incomplete similarity. In the first case the experimental data should cluster in the traditional $(\log\eta, \phi)$ plane $(\phi = u/u_*,\ \eta = u_*y/\nu = y/\delta)$ on the single straight line of the universal logarithmic law. In the second case the experimental points should occupy an area in the $(\log\eta, \phi)$ plane; to each value of the Reynolds number there corresponds a separate curve.

Our next hypothesis will be the *vanishing-viscosity principle:*

> **Second Hypothesis:** *A gradient of average velocity tends to a well-defined limit as the viscosity vanishes.*

This principle is in clear correspondence with the last part of Prandtl's conclusion, and was also implicitly used by von Kármán (see the Introduction).

The experiments, even at high Reynolds numbers, demonstrate a perceptible dependence of the velocity distribution on Re (see Figure 1). Therefore, and according to the vanishing viscosity principle, it is appropriate to expand $A(\mathrm{Re})$ and $\alpha(\mathrm{Re})$ into a series in a small parameter $\varepsilon(\mathrm{Re})$, vanishing at $\mathrm{Re} = \infty$, and to retain the first two terms

$$A = A_0 + A_1\varepsilon, \qquad \alpha = \alpha_0 + \alpha_1\varepsilon$$

where A_0, A_1, α_0 and α_1 should be, by the logic of the derivation, universal Reynolds-number-independent constants. We obtain from (27),

$$\partial_y u = \frac{u_*}{y}(A_0 + A_1\varepsilon)\left(\frac{y}{\delta}\right)^{\alpha_0+\alpha_1\varepsilon}. \tag{28}$$

When the viscosity (and, consequently, the length scale δ) tends to zero, a well-defined limit of (28) does exist for $\alpha_0 = 0$ only; therefore, according to our second hypothesis $\alpha_0 = 0$. There is also a possibility to assume that $A_0 = 0$, $\alpha_0 \neq 0$, and $\varepsilon = (\mathrm{Re})^{-\alpha_0}$. This leads to a universal power dependence of $\partial_y u$ upon y. This possibilitiy, however, is not found to be compatible with the experimental data.

Furthermore, (28) can be represented as

$$\begin{aligned}\partial_y u &= \frac{u_*}{y}(A_0 + A_1\varepsilon)\exp\left[\alpha_1\varepsilon\log\frac{y}{\delta}\right]\\ &= \frac{u_*}{y}(A_0 + A_1\varepsilon)\exp\left[\alpha_1\varepsilon\log\frac{u_* d}{\nu} + \alpha_1\varepsilon\log\frac{y}{d}\right].\end{aligned} \tag{29}$$

The small parameter ε is a function of the Reynolds number, vanishing at $\mathrm{Re} = \infty$. Relation (29) shows that, if ε tends to zero at $\mathrm{Re} \to \infty$ faster than $1/\log\mathrm{Re}$, the argument of the exponent tends to zero, and we return to the case of complete similarity. The experiments, as was mentioned before, show that this is not the case. If ε tends to zero slower than $1/\log\mathrm{Re}$, the well-defined limit of the velocity gradient for the viscosity going to zero does not exist, and we obtain a contradiction to the second hypothesis, the vanishing-viscosity principle. *Therefore, the only choice compatible with our basic hypotheses (incomplete similarity and vanishing viscosity principle)* is

$$\varepsilon = \frac{1}{\log\mathrm{Re}}. \tag{30}$$

Thus we obtain, by integration of (29)

$$\phi = \frac{u}{u_*} = (C_0\log\mathrm{Re} + C_1)\left(\frac{y}{\delta}\right)^{\alpha_1/\log\mathrm{Re}}. \tag{31}$$

Here an additional condition

$$\phi(0) = 0$$

was used. This condition is an independent assumption confirmed by experiments which does not follow from the no-slip boundary condition $u(0) = 0$, because the boundary $y = 0$ is outside the range of applicability of the intermediate asymptotic relation (29).

Thus, we come to a conclusion which is in correspondence with the intuitive idea of Prandtl (see the last sentence of his comment). Indeed, the *wall streaks where turbulence and viscosity act together penetrate the main body of the flow at any Reynolds number.* It is also clear that these streaks create the *intermittency* of the wall-bounded flows. This conclusion is also in correspondence with the idea of incomplete similarity.

The parameter of turbulence (u_*) and viscosity (ν) form together a monomial

$$C = (C_0 \log \mathrm{Re} + C_1) u_*^{1+(\alpha_1/\log \mathrm{Re})} \nu^{-(\alpha_1/\log \mathrm{Re})}, \tag{32}$$

whose dimension cannot be obtained from dimensional analysis, and it determines the velocity distribution

$$u = C y^{(\alpha_1/\log \mathrm{Re})}.$$

A careful comparison with the data of Nikuradze's experiments, which were performed under the direct guidance of L. Prandtl, suggested (see the details in [28–31]) the following values of the universal constants:

$$C_0 = \frac{1}{\sqrt{3}}, \qquad C_1 = \tfrac{5}{2}, \qquad \alpha_1 = \tfrac{3}{2}. \tag{33}$$

Therefore the *ultimate scaling law proposed for the velocity distribution in the major, intermediate region of the pipe is*

$$\phi = \left(\frac{\sqrt{3} + 5\alpha}{2\alpha} \right) \eta^{\alpha}, \qquad \alpha = \frac{3}{2 \log \mathrm{Re}}, \qquad \phi = \frac{u}{u_*}, \qquad \eta = \frac{u_* y}{\nu}. \tag{34}$$

The scaling law (34) shows, as expected, that there is no universal Re-independent velocity distribution in the $\log \eta, \phi$ plane, but there is a family of curves in this plane with Re as a parameter. However the family (34) has a special property of self-similarity and therefore of universality. Indeed, if we plot on the ordinate axis instead of ϕ the quantity

$$\psi = \frac{1}{\alpha} \log \frac{2\alpha\phi}{\sqrt{3} + 5\alpha}, \qquad \alpha = \frac{3}{2 \log \mathrm{Re}}, \tag{35}$$

we obtain $\psi = \log \eta$, *i.e.* the bisectrix of the first quadrant. Comparison with Nikuradze's experimental data shows that this is indeed the case (Figure 2). The overwhelming majority of the experimental points for $\eta > 30$ does indeed settle down to the bisectrix. The points corresponding to $\eta < 30$ naturally deviate from the bisectrix, but it should be emphasized that this deviation is a systematic one, not a scatter. We will discuss this topic in more detail elsewhere.

The scaling law (34) allows determination of the dependence of the drag coefficient on the Reynolds number. We define the dimensionless skin friction (drag) coefficient in a way now common in the literature (see, *e.g.* [17, p. 301])

$$\lambda = \frac{\tau}{\rho \bar{u}^2/8} = 8 \left(\frac{u_*^2}{\bar{u}^2} \right). \tag{36}$$

Note that von Kármán's definition of the skin-friction coefficient is different:

$$\lambda_K = \frac{2\tau}{\rho u_{\max}^2},$$

where $u_{\max}$ is the maximum velocity, so that according to (34),

$$\lambda_K = \lambda \frac{1}{(1+\alpha)^2 (2+\alpha)^2}. \tag{37}$$

The numerical factor $K = \lambda_K/\lambda$ is a function of the Reynolds number; at large Reynolds numbers K is close to $\frac{1}{4}$.

Using for the determination of the average velocity $\bar{u}$ the scaling law (34), and neglecting the deviation of the velocity distribution from the scaling law in the viscous sublayer, and near the axis, we obtain the formula for the *skin friction coefficient as a function of the Reynolds number*

$$\lambda = \frac{8}{\Psi^{2/(1+\alpha)}}, \qquad \Psi = \frac{e^{3/2}(\sqrt{3}+5\alpha)}{2^{\alpha}\alpha(1+\alpha)(2+\alpha)}, \qquad \alpha = \frac{3}{2\log \mathrm{Re}}. \tag{38}$$

Comparison of this law with the independent series of Nikuradze's experiments [9] determining the skin friction also showed an instructive agreement (see the details in [28–31]). The deviations are within the limits of a normal experimental scatter.

We come to the conclusion that the scaling law with the universal constants (33) and the drag law (38) describes the flow in smooth pipes satisfactorily for large Reynolds numbers, and that the incomplete similarity of this flow can be considered as established.

5. Modification of the Izakson–Millikan–von Mises derivation of the velocity distribution in the intermediate region. The vanishing-viscosity asymptotics

The universal logarithmic law hardened into dogma and became one of the pillars of turbulence theory and a mainstay of engineering science to a large extent because it was supported by an independent mathematical derivation based on seemingly unassailable principles. This derivation was proposed by Izakson [32], Millikan [33], and von Mises [34] (see also [17], pp. 299–301). It is usually presented as follows. It is assumed that in the intermediate region under consideration the dimensionless velocity distribution is a universal, Reynolds-number-independent function of the local Reynolds number $\eta = u_* y/\nu$. Thus, the influence of the external dimensional parameter, the pipe diameter d (and consequently of the Reynolds number) is neglected, so that the *wall law* is valid

$$\phi = \frac{u}{u_*} = f\left(\frac{u_* y}{\nu}\right), \tag{39}$$

where f is a certain dimensionless function.

On the other hand, in the vicinity of the pipe axis the *defect law* is assumed to be valid

$$u_{CL} - u = u_* g(2y/d), \tag{40}$$

where u_{CL} is the mean velocity at the pipe axis, so that $u_{CL} - u$ is the velocity defect, and g is another dimensionless function, but of a different argument: in relation (40) the influence of viscosity is neglected. Thus, it is assumed that in the wall region the influence of the external length scale d can be neglected, whereas near the pipe axis the influence of the internal length scale $\delta = \nu/u_*$ can be neglected. The next step is the assumption that there exists at very large Reynolds numbers an interval of distances where both laws (39) and (40) are valid. Therefore a functional equation

$$u_{CL} - u_* f(u_* y/\nu) = u_* g(2y/d) \tag{41}$$

is obtained by combining (39) and (40). After differentiation of (41) by y followed by multiplication by y, the following relation is obtained

$$\eta f'(\eta) = -\xi g'(\xi). \tag{42}$$

Here $\xi = 2y/d$. The right- and left-hand sides of Equation (42) contain functions of different arguments; therefore each of the sides can be only a constant. Denoting this constant by $1/\kappa$ and integrating, one obtains the law of the wall in the form of universal logarithmic law

$$f(\eta) = \frac{1}{\kappa} \log \eta + B, \tag{43}$$

as well as the defect law

$$g(\xi) = -\frac{1}{\kappa} \log \xi + B_0, \tag{44}$$

where

$$B_0 = \frac{u_{CL}}{u_*} - \frac{1}{\kappa} \log \frac{u_* d}{2\nu} - B. \tag{45}$$

This derivation was appreciated by A. N. Kolmogorov, and entered the survey of Sir James Lighthill [3, the relations (15)–(17) on p. 116]. It was apparently one of the first applications of the method of matched asymptotic expansions which is very popular nowadays (see the remarkable monographs [35, 36]).

This very attractive derivation is, however, not quite correct and needs a modification. It is clear now (*cf.* Figure 1) that both in the wall law and in the defect law the influence of Reynolds number cannot be neglected. So, according to our basic concept, these laws should be represented in the form

$$\phi = \frac{u}{u_*} = f\left(\frac{u_* y}{\nu}, \mathrm{Re}\right), \tag{46}$$

and

$$u_{CL} - u = u_* g\left(\frac{2y}{d}, \mathrm{Re}\right). \tag{47}$$

The further derivation proceeds as before, the only difference being that κ is no longer a constant, but a certain function of the Reynolds number: $\kappa = \kappa(\mathrm{Re})$, and B is also a function of the Reynolds number.

Therefore, the law of the wall takes the form

$$\phi = \frac{u}{u_*} = \frac{1}{\kappa(\mathrm{Re})} \log \frac{u_* y}{\nu} + B(\mathrm{Re}), \tag{48}$$

where $\kappa(\mathrm{Re})$ and $B(\mathrm{Re})$ are certain unspecified functions.

It is essential that there is no contradiction between the scaling law (34) and law of the wall (48). It was demonstrated in [37, 38], where the vanishing-viscosity method developed by Chorin (see [39, 40]) was essentially used. The law (34) can be written in the form

$$\phi = \left(\frac{1}{\sqrt{3}} \log \mathrm{Re} + \frac{5}{2}\right) \exp\left(\frac{3 \log \eta}{2 \log \mathrm{Re}}\right). \tag{49}$$

Let the observation point be at a fixed distance y from the wall, definitely larger than a certain length Δ, for instance, the size of a gauge. Let also the pipe diameter and pressure gradient be fixed. One is not free to vary $\mathrm{Re} = \bar{u}d/\nu$ and $\eta = u_* y/\nu$ independently, because the viscosity ν appears in both. So, if ν is decreased by the experimenter as it was in the Princeton

Superpipe experiments [26, 27], whose basic idea was proposed by Brown [41], one considers flows at ever larger η at ever larger Re, in particular the lowest $\eta = u_* \Delta / \nu$ is increasing with decreasing viscosity. Consider now the ratio $3 \log \eta / 2 \log \mathrm{Re}$ which enters the scaling law in the form of (49). It can be represented in the form

$$\frac{3 \log \eta}{2 \log \mathrm{Re}} = \frac{3}{2} \left[\log \frac{\bar{u} d}{\nu} + \log \frac{y}{d} + \log \frac{u_*}{\bar{u}} \right] \frac{1}{\log \frac{\bar{u} d}{\nu}}. \tag{50}$$

However, the distance from the wall y lies in the fixed interval $\Delta < y < d/2$, and $\bar{u}/u_*$ can be shown to be of the order of $\log \mathrm{Re}$, so that $\log(u_*/\bar{u}) \sim \log \log \mathrm{Re}$, which is asymptotically small at very large Re. Therefore $3 \log \eta / 2 \log \mathrm{Re}$ is asymptotically close to $\frac{3}{2}$, and the quantity

$$1 - \log \eta / \log \mathrm{Re}$$

can be considered to be a small parameter, so that

$$\begin{aligned} \exp \left[\frac{3}{2} \frac{\log \eta}{\log \mathrm{Re}} \right] &\approx \exp \left[\frac{3}{2} - \frac{3}{2} \left(1 - \frac{\log \eta}{\log \mathrm{Re}} \right) \right] \\ &= \mathrm{e}^{3/2} \left[1 - \frac{3}{2} \left(1 - \frac{\log \eta}{\log \mathrm{Re}} \right) \right] = \mathrm{e}^{3/2} \left[\frac{3}{2} \frac{\log \eta}{\log \mathrm{Re}} - \frac{1}{2} \right]. \end{aligned} \tag{51}$$

This means that in the interval of interest $\Delta < y < d/2$ the power law (34) can be approximated by

$$\phi = \mathrm{e}^{3/2} \left(\frac{\sqrt{3}}{2} + \frac{15}{4 \log \mathrm{Re}} \right) \log \eta - \frac{\mathrm{e}^{3/2}}{2\sqrt{3}} \log \mathrm{Re} - \frac{5}{4} \mathrm{e}^{3/2}, \tag{52}$$

i.e., by the relation of the form (48) with

$$\kappa(\mathrm{Re}) = \frac{\mathrm{e}^{-3/2}}{\frac{\sqrt{3}}{2} + \frac{15}{4 \log \mathrm{Re}}}, \qquad B = -\frac{\mathrm{e}^{3/2} \log \mathrm{Re}}{2\sqrt{3}} - \frac{5}{4} \mathrm{e}^{3/2}. \tag{53}$$

It is important that, for $\mathrm{Re} \to \infty$, the value of $\kappa(\mathrm{Re})$ tends to a finite nonzero limit $2/\sqrt{3}\, \mathrm{e}^{3/2} \simeq 0{\cdot}2776$, whereas the additive constant B, which has no finite limit, tends to $-\infty$.

At the same time, the family of power laws (34), having Re as the parameter, has an envelope (*cf*. Chorin's example presented in Section 3). The relation for the envelope is obtained in implicit form by elimination of Re from Equation (34) and the equation

$$\frac{3 \log \eta}{2 \log \mathrm{Re}} = \frac{\sqrt{3}}{10} \log \eta \left[\left(1 + \frac{20}{\sqrt{3} \log \eta} \right)^{1/2} - 1 \right], \tag{54}$$

which is obtained from (34) by its differentiation with respect to Re. And the envelope has an important feature: in the working range of $\log \eta$ it is practically indistinguishable (see [29, 31]) from the straight line

$$\phi = \frac{u}{u_*} = \frac{\sqrt{3} e}{2} \log \eta + 5{\cdot}1. \tag{55}$$

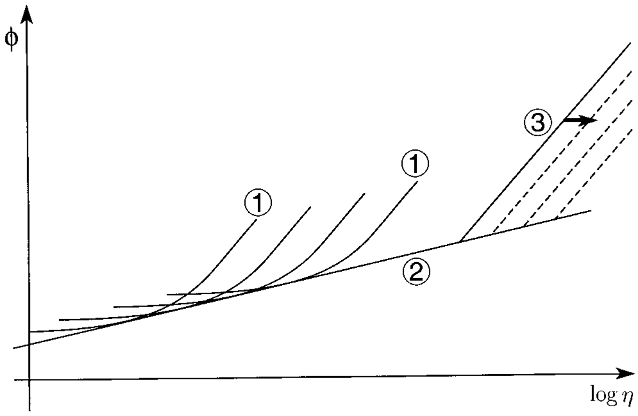

Figure 3. A schematic of the power-law curves in a pipe, their envelope and their asymptotic slope. The apparent motion of the curves to the right is due to the changes in Reynolds number. (1) The velocity as a function of the distance to the wall (in appropriate units), (2) the envelope of the power laws (formerly mistaken for the curves themselves), (3) the asymptotic rectilinear part of the law of the wall curves.

Bearing in mind that $2/\sqrt{3}e = 0{\cdot}425$, the straight line (55) can be identified with the traditional form of the universal logarithmic law (see *e.g.* [17], p. 273). Therefore, if one plots the experimental points that correspond to various values of Re on a single graph in the $(\log\eta, \phi)$-plane, what is natural for those who happen to believe the universal logarithmic law, the envelope will be revealed. The visual impact of the envelope, when plotting the experimental data in the $(\log\eta, \phi)$-plane, is magnified by the fact that the measurement at very small values of y, where the difference between the power laws and the envelope can also be noticed, is missing due to the experimental difficulties. Thus, if our proposed scaling law (34) is valid, the seeming confirmation of the universal logarithmic law is nothing but an illusion. The characteristic feature of the Reynolds-number-dependent scaling law, in addition to the splitting of the curves according to their Reynolds number, is the availability of straight-line parts at very large Reynolds numbers and the discrepancy of about $\sqrt{\mathrm{e}}$ between the slopes of the curves and the slope of the envelope (Figure 3). This qualitative distinction is confirmed by the experiments of the Princeton group [26, 27] (see Figure 1). Indeed, despite a flaw in these experiments discussed in detail in [31], the results of these experiments are sufficiently robust to exhibit a separate curve for each Reynolds number and a well-defined angle between the rectilinear parts of curves and their envelope.

6. Turbulent boundary layers

The universal law for the pipe flow (24) can be represented in dimensionless form

$$\eta\partial_\eta\phi = \frac{1}{\kappa} \tag{56}$$

and the Reynolds-number-dependent scaling law in a corresponding form

$$\eta\partial_\eta\phi = \left(\frac{\sqrt{3}}{2} + \frac{15}{4\log\mathrm{Re}}\right)\eta^{3/2\log\mathrm{Re}}. \tag{57}$$

The laws (25), (34) for the dimensionless velocity ϕ are obtained from (56) and (57) by integration.

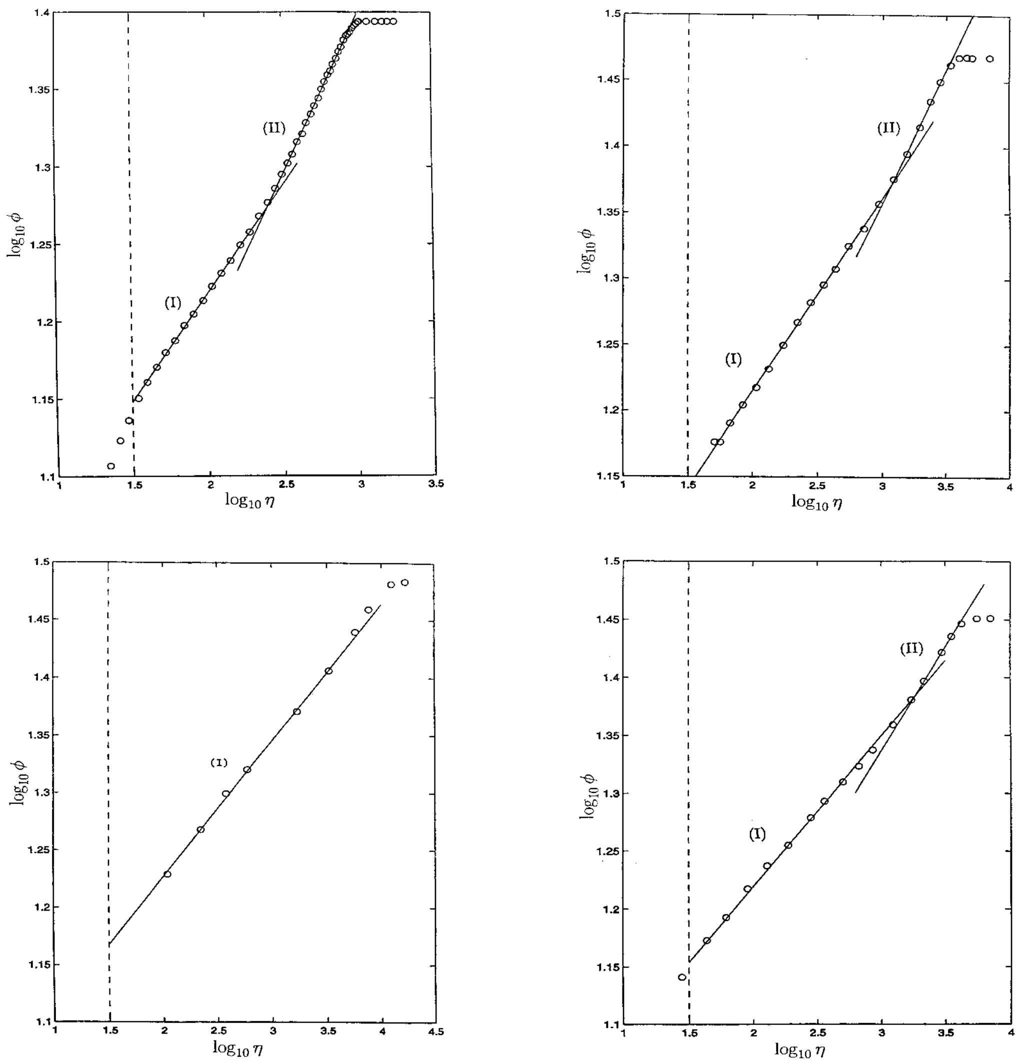

Figure 4. (a) The experiments by Erm and Joubert [44]. $\mathrm{Re}_\theta = 2\,788$. Both self-similar intermediate regions (I) and (II) are clearly seen. (b) The experiments of Krogstad and Antonia [45]. $\mathrm{Re}_\theta = 12\,570$. Both self-similar intermediate regions (I) and (II) are clearly seen. (c) The experiments of Petrie, Fontaine, Sommer and Brugart obtained by scanning the graphs in the review by Fernholz and Finley [46]. $\mathrm{Re}_\theta = 35\,530$. The first self-similar region (I) is revealed; the second self-similar region is not revealed. (d) The experiments of Smith obtained by scanning the graphs in the review by Fernholz and Finley [46]. $\mathrm{Re}_\theta = 12\,990$. The first self-similar intermediate region (I) is clearly seen; the second region (II) can be revealed.

By the logic of its derivation, the scaling law (57) should be valid, not only for the flow in pipes, but also for any wall-bounded shear flows.

Here, however, a basic question appears – what is the definition of the Reynolds number for these flows which allows use of law (34) for them? This basic question is immaterial as long as the engineer or researcher continues to believe in the universal logarithmic law.

Indeed, if the law is Re-independent, the definition of Re does not matter.[9] The situation is different when the law is Re-dependent. Indeed the laws (34) and (57) have the property of asymptotic covariance. This means that, if we replace Re based on, say, diameter by Re based on a different length scale, so that $\mathrm{Re} = Z\,\mathrm{Re}'$, where Z is a constant, the law (57), with accuracy to terms of the second order, will take the form

$$\eta\partial_\eta\phi = \left(\frac{\sqrt{3}}{2} + \frac{15}{4\log \mathrm{Re}'}\left(1 - \frac{Z}{\log \mathrm{Re}'}\right)\right)\eta^{3/(2\log \mathrm{Re}'),(1-(\log Z/\log \mathrm{Re}'))}. \tag{58}$$

Asymptotically, at $\log \mathrm{Re} \to \infty$, when the second-order terms can be neglected, (57) and (56) coincide, but practically, for large but not too large $\log \mathrm{Re}$ the terms can be significant. In this sense the choice of $\mathrm{Re} = \bar{u}d/\nu$ for pipe flows was lucky because it allowed neglect of the second-order terms, and use of the law of the wall, describing the velocity distribution in the intermediate region in the form of (34), (57). But what to do for other shear flows?

We will consider below the zero-pressure-gradient boundary layers, and we will show that the law (34) also describes these flows under appropriate choice of the Reynolds numbers. Zero-pressure-gradient boundary layers have been well investigated experimentally over the last 25 years. The common choice of Reynolds numbers for these flows is

$$\mathrm{Re}_\theta = \frac{U\theta}{\nu}, \tag{59}$$

where U is the free-stream velocity, and θ the momentum displacement thickness. This choice is rather arbitrary, and the law (34) with $\mathrm{Re} = \mathrm{Re}_\theta$ should not be valid for not extremely large Re_θ. But what is the proper choice of Re for the boundary layers?

To understand this we first of all have to confirm that in the intermediate layer of the boundary-layer flow adjacent to the viscous sublayer a certain scaling law is valid. To do that (see details in [42, 43]) we replotted all (available to us) experimental data presented in the traditional $(\log\eta, \phi)$-plane in a bilogarithmic plane $(\log_{10}\eta, \log_{10}\phi)$. The result was instructive: without exception for all investigated flows a straight line was obtained for region (I) adjacent to the viscous sublayer (see examples in Figure 4, and all details in [43]). Moreover, for the flows with low free-stream turbulence the second self-similar region (II) was obtained between the first one and the free-stream flow. The analysis of this region is beyond the scope of the present paper, but its degradation and subsequent disappearance with growing free-stream turbulence is proved persuasively by experiments of Hancock and Bradshaw [47]. The straight line (I) corresponds to a scaling law

$$\phi = A\eta^\alpha, \tag{60}$$

and coefficients A and α were obtained by statistical processing.

We assume that the effective Reynolds number Re has the form $\mathrm{Re} = U\Lambda/\nu$, where, we repeat, U is the free-stream velocity and Λ is a certain length scale. The basic question is whether such a unique length scale Λ, which plays the same role for the intermediate region (I) of the boundary layer as the diameter for the pipe flow, does exist? In other words, we must ask whether it is possible to find such a length scale Λ, perhaps influenced by individual

9 If you ask a barman to serve water without syrup, the question ‘without which syrup’ is inappropriate. But if the water is assumed to be with syrup, then the question ‘which syrup should be served’ is clear. The same situation as here.

Table 1.

Re_θ	α	A	$\log \mathrm{Re}_1$	$\log \mathrm{Re}_2$	$\log \mathrm{Re}$	$\mathrm{Re}_\theta/\mathrm{Re}$
Erm and Joubert [44]						
697	0·163	7·83	9·23	9·20	9·22	0·07
1 003	0·159	7·96	9·46	9·43	9·45	0·08
1 568	0·156	7·99	9·51	9·62	9·56	0·11
2 226	0·148	8·28	10·01	10·14	10·07	0·09
2 788	0·140	8·66	10·67	10·71	10·69	0·06
Krogstad and Antonia [45]						
12 570	0·146	8·38	10·18	10·27	10·23	0·45
Petrie, Fontaine, Sommer and Brungart[10]						
35 530	0·119	9·76	12·57	12·61	12·59	0·12
Smith[11]						
4 996	0·146	8·36	10·15	10·27	10·21	0·18
12 990	0·130	9·08	11·40	11·54	11·47	0·14

features of the flow, such that the scaling law (34) is valid for the first intermediate region (I). To answer this question we have taken the values A and α, obtained, we emphasize, by statistical processing of the experimental data in the first intermediate scaling region, and then calculated two values $\log \mathrm{Re}_1$, $\log \mathrm{Re}_2$, by solving the equations suggested by scaling law (34)

$$\frac{1}{\sqrt{3}} \log \mathrm{Re}_1 + \tfrac{5}{2} = A, \qquad \frac{3}{2 \log \mathrm{Re}_2} = \alpha. \tag{61}$$

If these values $\log \mathrm{Re}_1$, $\log \mathrm{Re}_2$ obtained by solving different Equations (61) are indeed close, *i.e.*, if they coincide with experimental accuracy, it means that the unique length scale Λ can be determined and the experimental scaling law in region (I) coincides with the basic scaling law (34).

Table 1 (above) shows in several examples that the values of $\log \mathrm{Re}_1$ and $\log \mathrm{Re}_2$ are close (more detailed discussion of processed data can be found in [43], but the conclusion remains the same). So, we can introduce for all these flows the mean Reynolds number

$$\mathrm{Re} = \sqrt{\mathrm{Re}_1 \, \mathrm{Re}_2} \qquad \log \mathrm{Re} = \tfrac{1}{2}(\log \mathrm{Re}_1 + \log \mathrm{Re}_2) \tag{62}$$

and consider Re as an estimate for the effective Reynolds number of the boundary-layer flow. Naturally the ratio $\mathrm{Re}_\theta/\mathrm{Re} = \theta/\Lambda$ is different for different flows.

Checking the universal form of the scaling law (34)

$$\psi = \frac{1}{\alpha} \log \left(\frac{2\alpha\phi}{\sqrt{3} + 5\alpha} \right) = \log \eta \tag{63}$$

we have another way of demonstrating clearly the applicability of scaling law (34) to the first intermediate region of the flow adjacent to the viscous sublayer. According to relation (63),

[10,11] The data obtained by scanning the graphs in the review [46].

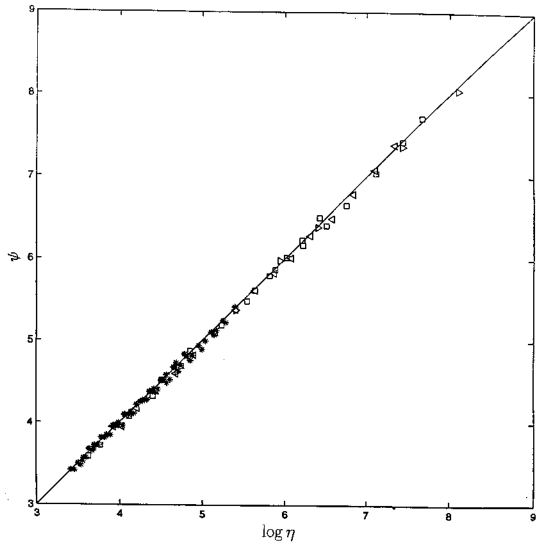

Figure 5. The experiments by Erm and Joubert [44] (∗). Krogstad and Antonia [45] (◁); Smith (□); and Petrie *et al.* (▷) collapse on the bisectrix of the first quadrant in accordance with the universal form (63) of the scaling law (34).

in the coordinates $(\log \eta, \psi)$, all experimental points should collapse onto the bisectrix of the first quadrant. In Figure 5 are represented the data of Erm and Joubert, [44], Krogstad and Antonia [45], Smith (data obtained by scanning the graphs in review [46]) and Petrie *et al.* (data obtained by scanning the graphs in review [46]). It is seen that the data collapse onto the bisectrix with sufficient accuracy to confirm the scaling law (34). The parameter α was calculated according to the formula $\alpha = (\frac{3}{2}\log \mathrm{Re})$, where here $\log \mathrm{Re}$ was taken to be $(\log \mathrm{Re}_1 + \log \mathrm{Re}_2)/2$ (see Table 1).

We conclude that scaling law (34) gives an accurate description of the mean velocity distribution over the self-similar intermediate region adjacent to the viscous sublayer for a wide variety of zero-pressure-gradient boundary-layer flows. The Reynolds number is defined as $\mathrm{Re} = U\Lambda/\nu$, where U is the free-stream velocity and Λ is a length scale which is well defined for all the flows under investigation.

The validity of the scaling law for boundary-layer flows constitutes a strong argument in favor of its validity for a wide class of wall-bounded turbulent shear flows at large Reynolds numbers.

7. Conclusion

Mathematics is still unable to obtain laws describing shear-flow turbulence from first principles. Properly directed experiments and analysis of experimental data remain crucially important methods for the mathematical modeling of turbulent flows. Combining theoretical fluid dynamics and the analysis of experimental data we have come to the conclusion that, contrary to basic hypothesis of von Kármán, Prandtl and their followers, the influence of viscosity never disappears, even at very large Reynolds-numbers. Thus, the general principle of Reynolds-number similarity, widely used in the literature, is not quite correct, and should be abandoned, as well as the universal logarithmic law. Our studies suggest instead the Reynolds-number-dependent scaling law discussed above. This law is based on incomplete similarity, so that viscosity enters the basic laws, but only in power combination with other parameters of turbulent motion.

We feel that the affirmation of the effectiveness of incomplete similarity and of vanishing-viscosity asymptotics for turbulent shear flows at large Reynolds numbers has broad implications for other manifestations of turbulence, *e.g*. in jets, wakes, mixing layers, and local structure, and should lead to a reconsideration of the basic tools used in the study of turbulent flows.

Sir James Lighthill knew this work in its evolution, practically from the very beginning. The last time the present author and A. J. Chorin delivered lectures concerning this problem in the presence of Sir James was in Rome in early July 1997. We clearly understood that our basic conclusion concerning the replacement of the universal logarithmic law by the Reynolds-number-dependent scaling law was expected by him and indeed, as he said, he came to agree with this conclusion after our first steps and when many things remained unclear for us. The attention and support of this giant was and will be a great stimulus in our work.

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